

GEOMETRIC CONSTRUCTION OF CRYSTAL BASES FOR QUANTUM GENERALIZED KAC-MOODY ALGEBRAS

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ABSTRACT. We provide a geometric realization of the crystal $B(\infty)$ for quantum generalized Kac-Moody algebras in terms of the irreducible components of certain Lagrangian subvarieties in the representation spaces of a quiver.

INTRODUCTION

There is a well-known and very fruitful interaction between the structure theory of quantum groups on the one hand and the geometry of quiver representations on the other. In the late 80's, Ringel realized the positive part $U_q^+(\mathfrak{g})$ of the quantized enveloping algebra of a Kac-Moody algebra \mathfrak{g} in the Hall algebra of any quiver whose underlying graph is the Dynkin diagram of \mathfrak{g} ([18]). This was soon followed by Lusztig's geometric construction of the *canonical basis* \mathbf{B} for $U_q^+(\mathfrak{g})$ in terms of simple perverse sheaves on the moduli spaces \mathcal{M}_α of representations of quivers ([13]). The combinatorial structure of this canonical basis is encoded in a colored graph $B(\infty)$, the *crystal graph* of $U_q^-(\mathfrak{g})$, whose vertices are the elements of \mathbf{B} ([9]).

By studying the cotangent geometry of \mathcal{M}_α , Kashiwara and Saito later gave a geometric construction of the crystal graph $B(\infty)$ ([11]). More precisely, they considered a certain Lagrangian subvariety $\mathcal{N}_\alpha \subset T^*\mathcal{M}_\alpha$ (first introduced in [14]) and built a graph \mathcal{B} whose vertices are the irreducible components of $\bigsqcup_\alpha \mathcal{N}_\alpha$ and whose arrows correspond to various generic fibrations between irreducible components of \mathcal{N}_α for different values of α . Using a combinatorial characterization of $B(\infty)$ in terms of tensor products with *elementary crystals*, the authors of [11] identified \mathcal{B} with $B(\infty)$. This work was later generalized by Saito who realized the crystals of all highest weight integrable

¹This research was supported by KRF Grant # 2007-341-C00001.

²This research was partially supported by Grant-in-Aid for Scientific Research (B) 18340007, Japan Society for the Promotion of Science.

representations using Lagrangian subvarieties in Nakajima's quiver varieties (see [19], [17]).

In another direction, Borchers was led in his study of the Moonshine and the Monster group to consider a new class of infinite-dimensional Lie algebras, now called the *generalized Kac-Moody algebras* ([1]). Although similar to Kac-Moody algebras in several respects, generalized Kac-Moody algebras allow for *imaginary* simple roots and play an important role in several different areas of mathematics (see, for example, [1, 2, 3, 7, 16]). The notion of the quantized enveloping algebra of a generalized Kac-Moody algebra \mathfrak{g} was defined in [6]. Several important structural properties of quantum groups were shown to exist also in this generalized setting. For instance, the integrable highest weight modules over \mathfrak{g} can be deformed to those over $U_q(\mathfrak{g})$ in such a way that the dimensions of weight spaces are invariant under the deformation.

In [4], the crystal basis theory was developed for quantum generalized Kac-Moody algebras and in [5], the notion of *abstract crystals* was introduced. Moreover, in [5], the authors proved a crystal embedding theorem and gave a characterization of the highest weight crystals $B(\infty)$ and $B(\lambda)$ for quantum generalized Kac-Moody algebras.

This paper is part of the project to extend the approach based on the geometry of quiver representations to the case of generalized Kac-Moody algebras and their quantized enveloping algebras. A (partly conjectural) geometric construction of the canonical basis \mathbf{B} of $U_q^+(\mathfrak{g})$ for an (even) generalized Kac-Moody algebra \mathfrak{g} was given in [8] (see also [15] and [12]). At the level of quivers, moving from Kac-Moody algebras to generalized Kac-Moody algebras means that one must now consider the quivers with edge loops and *semisimple* rather than simple perverse sheaves. In the present work, we provide an analogue of the construction in [11] of the crystal $B(\infty)$. Namely, we consider a certain Lagrangian subvariety $\mathcal{N}_\alpha \subset T^*\mathcal{M}_\alpha$, and build a crystal graph out of its irreducible components and generic fibrations among them. The noticeable difference with [11] is that the fibrations in question correspond, at the level of the representation theory of quivers, to extensions by *nonrigid* simple objects; i.e., the objects with non-vanishing self Ext^1 — typically a simple object sitting at a vertex with edge loops. We get around this difficulty by restricting our Lagrangian variety to the cotangent bundle of a certain open subset of \mathcal{M}_α by imposing that certain arrows are *regular semisimple* (see Section 2).

The paper is organized as follows. In Section 1, we recall various definitions and results pertaining to the crystal basis theory for quantum generalized Kac-Moody algebras as developed in [5]. In particular, we recall the characterization of the crystal $B(\infty)$ in terms of strict crystal embeddings $B(\infty) \rightarrow B(\infty) \otimes B_i$ for $i \in I$ (see Theorem 1.7). Sections 2 and Section 3 are devoted to the definition and study of the Lagrangian variety \mathcal{N}_α . The crystal structure on the set \mathcal{B} of irreducible components of $\bigsqcup_\alpha \mathcal{N}_\alpha$ is described in Section 3 (see Theorem 3.5). Finally, in Section 4, we prove the crystal isomorphism $\mathcal{B} \cong B(\infty)$ by constructing strict crystal embeddings $\mathcal{B} \rightarrow \mathcal{B} \otimes B_i$ for all $i \in I$ (see Theorem 4.4).

1. ABSTRACT CRYSTALS

Let I be a finite or countably infinite index set and let $A = (a_{ij})_{i,j \in I}$ be a *symmetric even integral Borcherds-Cartan matrix*. That is, A satisfies: (i) $a_{ii} \in \{2, 0, -2, -4, \dots\}$ for all $i \in I$, (ii) $a_{ij} = a_{ji} \in \mathbf{Z}_{\leq 0}$ for $i \neq j$. We say that an index $i \in I$ is *real* if $a_{ii} = 2$ and *imaginary* if $a_{ii} \leq 0$. We denote by $I^{re} = \{i \in I; a_{ii} = 2\}$ and $I^{im} = \{i \in I; a_{ii} \leq 0\}$ the set of real indices and the set of imaginary indices, respectively.

A *Borcherds-Cartan datum* (A, P, Π, Π^\vee) consists of

- (i) a Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$,
- (ii) a free abelian group P , the *weight lattice*,
- (iii) $\Pi = \{\alpha_i \in P; i \in I\}$, the set of *simple roots*,
- (iv) $\Pi^\vee = \{h_i; i \in I\} \subset P^\vee := \text{Hom}(P, \mathbf{Z})$, the set of *simple coroots*

satisfying the following properties:

- (a) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,
- (b) Π is linearly independent,
- (c) for any $i \in I$, there exists $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $j \in I$.

We use the notation $Q = \bigoplus_{i \in I} \mathbf{Z}\alpha_i$ and $Q_+ = \sum_{i \in I} \mathbf{Z}_{\geq 0}\alpha_i$.

Let q be an indeterminate. For an integer $n \in \mathbf{Z}$, define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = \prod_{k=1}^n [k], \quad \begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[n]![m-n]!}.$$

For a Borcherds-Cartan datum (A, P, Π, Π^\vee) , the *quantum generalized Kac-Moody algebra* $U_q(\mathfrak{g})$ is defined to be the associated algebra over $\mathbf{Q}(q)$ with 1 generated by the

elements e_i, f_i ($i \in I$), q^h ($h \in P^\vee$) subject to the defining relations:

$$\begin{aligned}
 (1.1) \quad & q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee, \\
 & q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \text{for } h \in P^\vee, i \in I, \\
 & e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \quad \text{for } i, j \in I, \text{ where } K_i = q^{h_i}, \\
 & \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \text{if } i \in I^{re} \text{ and } i \neq j, \\
 & \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \quad \text{if } i \in I^{re} \text{ and } i \neq j, \\
 & e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \quad \text{if } a_{ij} = 0.
 \end{aligned}$$

We denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by the e_i 's (resp. the f_i 's).

We recall the notion of *abstract crystals* for quantum generalized Kac-Moody algebras introduced in [5].

Definition 1.1. An *abstract $U_q(\mathfrak{g})$ -crystal* or simply a *crystal* is a set B together with the maps $\text{wt}: B \rightarrow P$, $\tilde{e}_i, \tilde{f}_i: B \rightarrow B \sqcup \{0\}$ and $\varepsilon_i, \varphi_i: B \rightarrow \mathbf{Z} \sqcup \{-\infty\}$ ($i \in I$) satisfying the following conditions:

- (i) $\text{wt}(\tilde{e}_i b) = \text{wt } b + \alpha_i$ if $i \in I$ and $\tilde{e}_i b \neq 0$,
- (ii) $\text{wt}(\tilde{f}_i b) = \text{wt } b - \alpha_i$ if $i \in I$ and $\tilde{f}_i b \neq 0$,
- (iii) for any $i \in I$ and $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt } b \rangle$,
- (iv) for any $i \in I$ and $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,
- (v) for any $i \in I$ and $b \in B$ such that $\tilde{e}_i b \neq 0$, we have
 - (a) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $i \in I^{re}$,
 - (b) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b)$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + a_{ii}$ if $i \in I^{im}$,
- (vi) for any $i \in I$ and $b \in B$ such that $\tilde{f}_i b \neq 0$, we have
 - (a) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $i \in I^{re}$,
 - (b) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b)$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - a_{ii}$ if $i \in I^{im}$,
- (vii) for any $i \in I$ and $b \in B$ such that $\varphi_i(b) = -\infty$, we have $\tilde{e}_i b = \tilde{f}_i b = 0$.

Definition 1.2. Let B_1 and B_2 be crystals. A map $\psi: B_1 \rightarrow B_2$ is a *crystal morphism* if it satisfies the following properties:

(i) for $b \in B_1$, we have

$$\text{wt}(\psi(b)) = \text{wt}(b), \varepsilon_i(\psi(b)) = \varepsilon_i(b), \varphi_i(\psi(b)) = \varphi_i(b) \text{ for all } i \in I,$$

(ii) for $b \in B_1$ and $i \in I$ with $\tilde{f}_i b \in B_1$, we have $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

Definition 1.3. Let $\psi: B_1 \rightarrow B_2$ be a crystal morphism.

(a) ψ is called a *strict morphism* if

$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b), \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \text{ for all } i \in I \text{ and } b \in B_1.$$

Here, we understand $\psi(0) = 0$.

(b) ψ is called an *embedding* if the underlying map $\psi: B_1 \rightarrow B_2$ is injective.

We will often use the notation $\text{wt}_i(b) = \langle h_i, \text{wt}(b) \rangle$ ($i \in I, b \in B$).

Example 1.4. Fix $i \in I$. For any $u \in U_q^-(\mathfrak{g})$, there exist unique $v, w \in U_q^-(\mathfrak{g})$ such that

$$e_i u - u e_i = \frac{K_i v - K_i^{-1} w}{q_i - q_i^{-1}}.$$

We define the endomorphism $e'_i: U_q^-(\mathfrak{g}) \rightarrow U_q^-(\mathfrak{g})$ by $e'_i(u) = w$. Then every $u \in U_q^-(\mathfrak{g})$ has a unique *i-string decomposition*

$$u = \sum_{k \geq 0} f_i^{(k)} u_k, \quad \text{where } e'_i u_k = 0 \text{ for all } k \geq 0,$$

where

$$f_i^{(k)} := \begin{cases} f_i^k / [k]! & \text{if } i \text{ is real,} \\ f_i^k & \text{if } i \text{ is imaginary.} \end{cases}$$

The *Kashiwara operators* \tilde{e}_i, \tilde{f}_i ($i \in I$) are then defined by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

Let $\mathbf{A}_0 = \{f/g \in \mathbf{Q}(q); f, g \in \mathbf{Q}[q], g(0) \neq 0\}$ and let $L(\infty)$ be the \mathbf{A}_0 -submodule of $U_q^-(\mathfrak{g})$ generated by

$$\left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1}; r \geq 0, i_k \in I \right\},$$

where $\mathbf{1}$ is the multiplicative identity in $U_q^-(\mathfrak{g})$. Then the set

$$B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} + qL(\infty); r \geq 0, i_k \in I \right\} \setminus \{0\} \subset L(\infty)/qL(\infty)$$

becomes a crystal with the maps wt , \tilde{e}_i , \tilde{f}_i , ε_i , φ_i ($i \in I$) defined by

$$\begin{aligned} \text{wt}(b) &= -(\alpha_{i_1} + \cdots + \alpha_{i_r}) \quad \text{for } b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} + qL(\infty), \\ \varepsilon_i(b) &= \begin{cases} \max \{k \geq 0; \tilde{e}_i^k b \neq 0\} & \text{for } i \in I^{\text{re}}, \\ 0 & \text{for } i \in I^{\text{im}}, \end{cases} \\ \varphi_i(b) &= \varepsilon_i(b) + \text{wt}_i(b) \quad (i \in I). \end{aligned}$$

Example 1.5. For each $i \in I$, let $B_i = \{b_i(-n); n \geq 0\}$. Then B_i is a crystal with the maps defined by

$$\begin{aligned} \text{wt } b_i(-n) &= -n\alpha_i, \\ \tilde{e}_i b_i(-n) &= b_i(-n+1), \quad \tilde{f}_i b_i(-n) = b_i(-n-1), \\ \tilde{e}_j b_i(-n) &= \tilde{f}_j b_i(-n) = 0 \quad \text{if } j \neq i, \\ \varepsilon_i(b_i(-n)) &= n, \quad \varphi_i(b_i(-n)) = -n \quad \text{if } i \in I^{\text{re}}, \\ \varepsilon_i(b_i(-n)) &= 0, \quad \varphi_i(b_i(-n)) = \text{wt}_i(b_i(-n)) = -na_{ii} \quad \text{if } i \in I^{\text{im}}, \\ \varepsilon_j(b_i(-n)) &= \varphi_j(b_i(-n)) = -\infty \quad \text{if } j \neq i. \end{aligned}$$

Here, we understand $b_i(-n) = 0$ for $n < 0$. The crystal B_i is called an *elementary crystal*.

For a pair of crystals B_1 and B_2 , their *tensor product* is defined to be the set

$$B_1 \otimes B_2 = \{b_1 \otimes b_2; b_1 \in B_1, b_2 \in B_2\},$$

where the crystal structure is defined as follows: The maps $\text{wt}, \varepsilon_i, \varphi_i$ are given by

$$\begin{aligned} \text{wt}(b \otimes b') &= \text{wt}(b) + \text{wt}(b'), \\ \varepsilon_i(b \otimes b') &= \max(\varepsilon_i(b), \varepsilon_i(b') - \text{wt}_i(b)), \\ \varphi_i(b \otimes b') &= \max(\varphi_i(b) + \text{wt}_i(b'), \varphi_i(b')). \end{aligned}$$

For $i \in I$, we define

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'), \end{cases}$$

For $i \in I^{\text{re}}$, we define

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b'), \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases}$$

and, for $i \in I^{\text{im}}$, we define

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') - a_{ii}, \\ 0 & \text{if } \varepsilon_i(b') < \varphi_i(b) \leq \varepsilon_i(b') - a_{ii}, \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases}$$

We recall the *crystal embedding theorem* proved in [5].

Theorem 1.6. [5] *For each $i \in I$, there exists a unique strict crystal embedding*

$$\Psi_i: B(\infty) \rightarrow B(\infty) \otimes B_i$$

which sends $\mathbf{1}$ to $\mathbf{1} \otimes b_i(0)$.

As an application of the crystal embedding theorem, we obtain a characterization of the crystal $B(\infty)$.

Theorem 1.7. [5] *Let B be a crystal. Suppose that B satisfies the following conditions:*

- (i) $\text{wt}(B) \subset -Q_+$,
- (ii) *there exists an element $b_0 \in B$ such that $\text{wt}(b_0) = 0$,*
- (iii) *for any $b \in B$ such that $b \neq b_0$, there exists some $i \in I$ such that $\tilde{e}_i b \neq 0$,*
- (iv) *for each $i \in I$, there exists a strict crystal embedding $\Psi_i: B \rightarrow B \otimes B_i$.*

Then there is a crystal isomorphism

$$B \xrightarrow{\sim} B(\infty),$$

which sends b_0 to $\mathbf{1}$.

2. QUIVER VARIETY

Let (I, H) be a quiver with an orientation Ω . Namely, we have maps

$$\text{out}, \text{in}: H \rightarrow I$$

and an involution $-$ of H such that $\text{out}(\bar{h}) = \text{in}(h)$ for any $h \in H$. We assume that $-$ does not have a fixed point, and $H = \Omega \sqcup \bar{\Omega}$. If $i = \text{out}(h)$ and $j = \text{in}(h)$, then we say that h is an arrow from i to j and write $h: i \rightarrow j$. If $h \in H$ satisfies $\text{out}(h) = \text{in}(h)$, then we say that h is a loop. We denote by H^{loop} (resp. Ω^{loop}) the set of all loops in H (resp. in Ω). Let c_{ij} denote the number of arrows in H from i to j , and define

$$a_{ij} = \begin{cases} 2 - c_{ii} = 2 - (\text{the number of loops at } i \text{ in } H) & \text{if } i = j, \\ -c_{ij} = -(\text{the number of arrows in } H \text{ from } i \text{ to } j) & \text{if } i \neq j. \end{cases}$$

Then $A = (a_{ij})_{i,j \in I}$ becomes a symmetric even integral Borcherds-Cartan matrix.

For $\alpha \in Q_+$, let $V_\alpha = \bigoplus_{i \in I} V_i$ be an I -graded vector space with

$$\underline{\dim} V_\alpha := \sum_{i \in I} (\dim V_i) \alpha_i = \alpha,$$

let $GL_\alpha = \prod_i GL(V_i)$, and set

$$X_\alpha = \bigoplus_{h \in H} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}).$$

We introduce a symmetric bilinear form $(\ , \)$ on Q by the formula

$$\left(\sum_k d_k \alpha_k, \sum_j e_j \alpha_j \right) = \sum_k d_k e_k.$$

Thus $\dim X_\alpha = ((2 \text{Id} - A) \cdot \alpha, \alpha)$.

The symplectic form ω on X_α is given by

$$\omega(B, B') = \sum_h \varepsilon(h) \text{Tr}(B_{\bar{h}} B'_h),$$

where

$$\varepsilon(h) = \begin{cases} 1 & \text{if } h \in \Omega, \\ -1 & \text{if } h \in \bar{\Omega}. \end{cases}$$

Consider the *moment map* $\mu = (\mu_i: X_\alpha \rightarrow \text{End } V_i)_{i \in I}$ given by

$$\mu_i(B) = \sum_{\substack{h \in H \\ \text{out}(h)=i}} \varepsilon(h) B_{\bar{h}} B_h.$$

Let X_α° denote the set of B 's such that B_h is regular semisimple for all $h \in \bar{\Omega}^{\text{loop}}$. Then X_α° is a Zariski open subset of X_α . Let \mathcal{N}_α be the variety consisting of all $B = (B_h)_{h \in H} \in X_\alpha$ satisfying the following three conditions:

(i) there exists an I -graded complete flag $F = (F_0 \subset F_1 \subset F_2 \subset \cdots)$ such that

$$B_h(F_k) \subset F_k \text{ for all } h \in \overline{\Omega}^{loop}, \quad B_h(F_k) \subset F_{k-1} \text{ for all } h \in H \setminus \overline{\Omega}^{loop},$$

(ii) $\mu_i(B) = 0$ for all $i \in I$,

(iii) $B \in X_\alpha^\circ$.

Then \mathcal{N}_α is a Zariski closed subvariety of X_α° .

We first prove:

Lemma 2.1. *The variety \mathcal{N}_α is isotropic.*

Proof. We first recall the following general fact. Let X be a smooth algebraic variety, Y a projective variety, and Z a smooth closed algebraic subvariety of $X \times Y$. Consider the Lagrangian variety $\Lambda = T_Z^*(X \times Y)$ and the projection map $q: \Lambda \cap (T^*X \times T^*Y) \rightarrow T^*X$. Then it is known that the image of q is isotropic (e.g. see [10, Proposition 8.3]).

We apply this fact to the case where $X = \mathcal{M}_\alpha := \prod_{h \in \Omega} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)})$, the moduli space of representations of the quiver (I, Ω) , $Y = \mathfrak{B}$, the variety of I -graded complete flags, and

$$Z = \{((B_h)_{h \in \Omega}, F) \in X \times Y; B_h F_k \subset F_{k-1} \text{ for all } k \geq 0\}.$$

Then

$$\begin{aligned} T^*\mathfrak{B} = \{&(F, K); F \in \mathfrak{B}, K \text{ is an } I\text{-graded endomorphism of } V \\ &\text{such that } K(F_k) \subset F_{k-1} \text{ for all } k \geq 0\}, \end{aligned}$$

$$T^*\mathcal{M}_\alpha = X_\alpha,$$

$$\begin{aligned} \Lambda = \{&(B, F, K); K = \sum \varepsilon(h) B_h B_{\overline{h}}, \\ &B_h F_k \subset F_{k-1}, B_{\overline{h}} F_k \subset F_k \text{ for all } h \in \Omega, k \geq 0\} \end{aligned}$$

and

$$\begin{aligned} \text{Im } q = \{&B = (B_h)_{h \in H}; \text{ there exists } F \in \mathfrak{B} \text{ such that } B_h(F_k) \subset F_{k-1}, B_{\overline{h}}(F_k) \subset F_k \\ &\text{for all } h \in \Omega, k \geq 0, \mu_i(B) = 0 \text{ for all } i \in I\}. \end{aligned}$$

Since our variety \mathcal{N}_α is contained in $\text{Im } q$, which is isotropic, we are done. \square

Later we will show that \mathcal{N}_α is Lagrangian for all α . For the moment, we consider a simple case :

Lemma 2.2. *For any $l \geq 1$ and $i \in I$, the variety $\mathcal{N}_{l\alpha_i}$ is Lagrangian and irreducible.*

Proof. We have to prove that $\mathcal{N}_{l\alpha_i}$ is irreducible and $\dim(\mathcal{N}_{l\alpha_i}) = \frac{1}{2} \dim(X_\alpha)$.

If i is a real vertex, then $\mathcal{N}_{l\alpha_i} = X_{l\alpha_i} = \{pt\}$.

Assume now that i is an imaginary vertex and that $c_{ii} = 2$. Then $\Omega_i^{loop} = \{h\}$ and the moment map condition (ii) implies $[B_{\bar{h}}, B_h] = 0$. Note that $|\Omega_i^{loop}| = \frac{1}{2}c_{ii}$. Since $B_{\bar{h}}$ is regular semisimple and B_h is nilpotent, $B_h = 0$. Therefore $\mathcal{N}_{l\alpha_i} \simeq \mathfrak{gl}(l)^{reg}$, the set of regular semisimple elements of $\mathfrak{gl}(l)$. In particular, $\mathcal{N}_{l\alpha_i}$ is irreducible and of dimension $l^2 = \frac{1}{2} \dim X_{l\alpha_i}$.

Finally, consider the case where $c_{ii} > 2$, let \mathfrak{B} be the flag variety of $\mathfrak{gl}(l)$ and let

$$Y = \left\{ ((B_h, B_{\bar{h}})_{h \in \Omega}, \mathfrak{b}) ; B_{\bar{h}} \in \mathfrak{b}^{reg}, B_h \in \mathfrak{n}, \sum_{h \in \Omega} [B_{\bar{h}}, B_h] = 0 \right\},$$

where \mathfrak{b} is a Borel subalgebra containing all $B_h, B_{\bar{h}}$'s, \mathfrak{b}^{reg} is the set of regular semisimple elements in \mathfrak{b} and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Set

$$Y^\circ = \{ ((B_h, B_{\bar{h}})_{h \in \Omega}, \mathfrak{b}) \in Y ; B_h \text{'s are regular nilpotent for all } h \in \Omega \}.$$

Then Y° is an open dense subset of Y . Let $\pi : Y \rightarrow \mathfrak{B}$ be the natural projection. Because each $B_{\bar{h}}$ is regular simple, $\text{ad}(B_{\bar{h}}) : \mathfrak{n} \rightarrow \mathfrak{n}$ is invertible. It follows that $\pi^{-1}(\mathfrak{b})$ is a vector bundle over $(\mathfrak{b}^{reg})^{c_{ii}/2}$ of rank $(c_{ii}/2 - 1) \dim \mathfrak{n}$. Hence $\pi^{-1}(\mathfrak{b})$ is irreducible and of dimension $\frac{1}{2}c_{ii} \dim \mathfrak{b} + (\frac{1}{2}c_{ii} - 1) \dim \mathfrak{n} = \frac{1}{2}(l^2c_{ii} - l(l-1))$. Therefore, Y° is irreducible, of dimension $\frac{1}{2}(l^2c_{ii} - l(l-1)) + \dim \mathfrak{B} = \frac{1}{2}l^2c_{ii} = \frac{1}{2} \dim X_{l\alpha_i}$. Since there is a natural finite surjective map $Y \rightarrow \mathcal{N}_{l\alpha_i}$ whose restriction to Y° is injective, Y° can be regarded as an open dense subset of $\mathcal{N}_{l\alpha_i}$. Hence $\mathcal{N}_{l\alpha_i}$ is irreducible of dimension $\frac{1}{2} \dim X_{l\alpha_i}$ as desired. \square

Next, we introduce stratifications of \mathcal{N}_α (one for each $i \in I$) as follows. Fix $i \in I$ and let t be the number of loops at i in Ω . Let $\mathcal{R} = \mathbf{C}\langle x_1, \dots, x_t, y_1, \dots, y_t \rangle$ be the free associative algebra generated by x_i, y_i ($i = 1, \dots, t$). Write $\Omega_i^{loop} = \{\sigma_1, \dots, \sigma_t\}$.

For $B = (B_h)_{h \in H} \in \mathcal{N}_\alpha$ and $f \in \mathcal{R}$, we define

$$\begin{aligned} f(B) &= f(B_{\sigma_1}, \dots, B_{\sigma_t}, B_{\bar{\sigma}_1}, \dots, B_{\bar{\sigma}_t}), \\ \mathbf{C}\langle B \rangle_i &= \{f(B) ; f \in \mathcal{R}\}, \\ \varepsilon_i(B) &= \text{codim}_{V_i} \left(\mathbf{C}\langle B \rangle_i \cdot \sum_{\substack{h: j \rightarrow i \\ j \neq i}} \text{Im } B_h \right). \end{aligned}$$

We put $\mathcal{N}_{\alpha,n,i} = \{B \in \mathcal{N}_\alpha; \varepsilon_i(B) = n\}$. It is clear that this defines a finite stratification $\mathcal{N}_\alpha = \bigsqcup_{n \geq 0} \mathcal{N}_{\alpha,n,i}$ into locally closed subsets.

We choose an identification $V_i \xrightarrow{\sim} V_i^*$ and for $B \in X_\alpha$, we define $B^* \in X_\alpha$ by $(B^*)_h = B_h^t: V_{\text{out}(h)} \rightarrow V_{\text{in}(h)}$ for $h \in H$. The map $B \mapsto B^*$ defines an involution on X_α and \mathcal{N}_α . Set

$$\varepsilon_i^*(B) = \varepsilon_i(B^*) = \text{codim}_{V_i} \left(\mathbf{C}\langle B^t \rangle_i \cdot \sum_{\substack{h: i \rightarrow j \\ i \neq j}} \text{Im } B_h^t \right) = \dim \bigcap_{\substack{h: i \rightarrow j \\ i \neq j}} \text{Ker}(B_h \cdot \mathbf{C}\langle B \rangle_i).$$

Here $\text{Ker}(B_h \cdot \mathbf{C}\langle B \rangle_i) = \{v \in V_i; B_h \cdot \mathbf{C}\langle B \rangle_i v = 0\}$. Let $\text{Irr } \mathcal{N}_\alpha$ denote the set of all irreducible components of \mathcal{N}_α . Of course, the $*$ -involution induces an involution on $\text{Irr } \mathcal{N}_\alpha$ as well. It does not depend on the choice of an isomorphism $V \simeq V^*$, because \mathcal{N}_α is invariant by the action of GL_α .

For $\Lambda \in \text{Irr } \mathcal{N}_\alpha$, we define

$$\varepsilon_i(\Lambda) = \varepsilon_i(B), \quad \varepsilon_i^*(\Lambda) = \varepsilon_i^*(B),$$

where B is a generic point of Λ . By the definition, $\Lambda \subset \mathcal{N}_{\alpha, \geq n, i}$ and $\Lambda \cap \mathcal{N}_{\alpha, n, i}$ is dense in Λ if and only if $n = \varepsilon_i(\Lambda)$.

Lemma 2.3. *The following statements hold.*

- (a) $\varepsilon_i^*(\Lambda) = \varepsilon_i(\Lambda^*)$ for all $i \in I$.
- (b) If $\Lambda \in \text{Irr } \mathcal{N}_\alpha$ and $\varepsilon_i(\Lambda) = 0$ for all $i \in I$, then $\alpha = 0$ and $\Lambda = 0$.
- (c) If $\Lambda \in \text{Irr } \mathcal{N}_\alpha$ and $\varepsilon_i^*(\Lambda) = 0$ for all $i \in I$, then $\alpha = 0$ and $\Lambda = 0$.

Proof. Statement (a) is obvious from the definitions. By (a), statements (b) and (c) are equivalent. We now prove (b). Let Λ be as in (b), and assume that $\alpha \neq 0$. Let $B \in \Lambda$ be a generic point, so that $\varepsilon_i(B) = 0$ for all i . By condition (i), there exists an I -graded complete flag $F = (F_0 \subset F_1 \cdots \subset F_d)$ such that $B_h(F_k) \subset F_k$ for all h and k . In particular, F_{d-1} is stable under all operators B_h . Let $i_0 \in I$ be such that $\underline{\dim}(F_d/F_{d-1}) = \alpha_{i_0}$. We have $\mathbf{C}\langle B \rangle_{i_0} \cdot \sum_{\substack{h: j \rightarrow i_0 \\ j \neq i_0}} \text{Im } B_h \subset F_{d-1}$. But this yields $\varepsilon_{i_0}(B) \geq 1$, which is a contradiction. \square

3. CRYSTAL STRUCTURE

For $i \in I$, $l \in \mathbf{N}$ and $\alpha = \sum d_i \alpha_i \in Q_+$, let

$$\begin{aligned} E_{\alpha;l\alpha_i} &= \{(B, B', B'', \phi', \phi) ; B' \in \mathcal{N}_\alpha, B'' \in \mathcal{N}_{l\alpha_i}, B \in \mathcal{N}_{\alpha+l\alpha_i}, \\ &0 \longrightarrow V_\alpha \xrightarrow{\phi} V_{\alpha+l\alpha_i} \xrightarrow{\phi'} V_{l\alpha_i} \longrightarrow 0 \text{ is exact,} \\ &\phi \circ B' = B \circ \phi, \phi' \circ B = B'' \circ \phi'\} \end{aligned}$$

be the space parametrizing the extensions of representations of the quiver (I, H) . There are canonical maps

$$(3.1) \quad X_\alpha \times X_{l\alpha_i} \xleftarrow{p} E_{\alpha;l\alpha_i} \xrightarrow{q} X_{\alpha+l\alpha_i}$$

given by

$$p(B, B', B'', \phi, \phi') = (B', B''), \quad q(B, B', B'', \phi, \phi') = B$$

and we may project further $p_1: E_{\alpha;l\alpha_i} \longrightarrow X_\alpha$. Put $\mathcal{N}_{\alpha;l\alpha_i} = q^{-1}(\mathcal{N}_{\alpha+l\alpha_i})$. Then the diagram (3.1) restricts to

$$(3.2) \quad \begin{array}{ccccc} \mathcal{N}_\alpha \times \mathcal{N}_{l\alpha_i} & \xleftarrow{p} & \mathcal{N}_{\alpha;l\alpha_i} & \xrightarrow{q} & \mathcal{N}_{\alpha+l\alpha_i} \\ & \searrow & \downarrow p_1 & & \\ & & \mathcal{N}_\alpha & & \end{array}$$

Observe that p is not surjective in general. Indeed, if $i \in I$ is imaginary and $h \in \overline{\Omega}^{loop}$ is an edge loop at i , then for any $(B, B', B'', \phi, \phi') \in \mathcal{N}_{\alpha;l\alpha_i}$, the operator B_h is regular semisimple, which implies that the spectra of B'_h and B''_h are disjoint. Let us denote by $\mathcal{N}_\alpha \times^{reg} \mathcal{N}_{l\alpha_i} \subset \mathcal{N}_\alpha \times \mathcal{N}_{l\alpha_i}$ the open subset of pairs (B', B'') for which the operators B'_h, B''_h have disjoint spectra for any edge loop $h \in \overline{\Omega}^{loop}$ at i .

For any $n \geq 0$, we define $\mathcal{N}_{\alpha,n,i} \times^{reg} \mathcal{N}_{l\alpha_i}$ to be the intersection of $\mathcal{N}_\alpha \times^{reg} \mathcal{N}_{l\alpha_i}$ with $\mathcal{N}_{\alpha,n,i} \times \mathcal{N}_{l\alpha_i}$. The locally closed subspace $\mathcal{N}_{\alpha,n,i;l\alpha_i}$ of $\mathcal{N}_{\alpha;l\alpha_i}$ is defined in a similar fashion. That is, $\mathcal{N}_{\alpha,n,i;l\alpha_i} = q^{-1}(\mathcal{N}_{\alpha+l\alpha_i,n,i})$. Finally, let $Z_{\alpha,l\alpha_i}$ be the set of short exact sequences (of I -graded vector spaces) $0 \longrightarrow V_\alpha \xrightarrow{\phi} V_{\alpha+l\alpha_i} \xrightarrow{\phi'} V_{l\alpha_i} \longrightarrow 0$. Note that $GL_{\alpha+l\alpha_i}$ acts on $Z_{\alpha,l\alpha_i}$ transitively.

Proposition 3.1. *The following statements hold :*

- (a) *The restriction of q to $\mathcal{N}_{\alpha,l,i;l\alpha_i}$ is a $GL_\alpha \times GL_{l\alpha_i}$ -principal bundle.*

(b) For $\alpha = \sum_k d_k \alpha_k$, the restriction of p to $\mathcal{N}_{\alpha, l, i; l\alpha_i}$ factors as

$$\mathcal{N}_{\alpha, l, i; l\alpha_i} \xrightarrow{p'} (\mathcal{N}_{\alpha, 0, i} \times^{reg} \mathcal{N}_{l\alpha_i}) \times Z_{\alpha, l\alpha_i} \xrightarrow{p''} \mathcal{N}_{\alpha, 0, i} \times^{reg} \mathcal{N}_{l\alpha_i},$$

where p'' is the natural projection and p' is an affine fibration of rank

$$r = l \left(\sum_{j \neq i} c_{ij} d_j + (c_{ii} - 1) d_i \right) = (l\alpha_i, (\text{Id} - A) \cdot \alpha).$$

Proof. By the definition, if $(B, B', B'', \phi, \phi')$ belongs to $\mathcal{N}_{\alpha, l, i; l\alpha_i}$, then there exists a unique B -invariant submodule $W \subset V_{\alpha+l\alpha_i}$ such that $\dim(V_{\alpha+l\alpha_i}/W) = l\alpha_i$. Namely, W is the submodule generated by $\bigoplus_{k \neq i} V_k$. This means that $\text{Im}(\phi)$ is uniquely determined, and thus ϕ, ϕ' are also determined up to a (free) $GL_{\alpha} \times GL_{l\alpha_i}$ -action, which proves (a).

We turn to (b). The map p' is given by $(B, B', B'', \phi, \phi') \mapsto ((B', B''), (\phi, \phi'))$. Note that by the above argument the image of p' indeed lies in $(\mathcal{N}_{\alpha, 0, i} \times^{reg} \mathcal{N}_{l\alpha_i}) \times Z_{\alpha, l\alpha_i}$. Now let us fix (B', B'', ϕ, ϕ') , set $W = \phi(V_{\alpha})$ and choose a complement U to W in $(V_{\alpha+l\alpha_i})_i$. Thus $\dim U = l$ and $\dim W_i = d_i$. We identify $V_{l\alpha_i}$ with U via ϕ' , and V_{α} with $\text{Im}(\phi)$ via ϕ . The fiber of p' consists of operators $B = (B_h)_h \in \mathcal{N}_{\alpha+l\alpha_i}$ which restrict to B' on W and induces B'' on U . We may write

$$\begin{cases} B_h = B'_h & \text{if } \text{out}(h) \neq i, \\ B_h = B'_h + y_h & \text{if } \text{out}(h) = i, \text{ in}(h) = j \neq i, \text{ where } y_h: U \rightarrow (V_{\alpha})_j, \\ B_h = B'_h + B''_h + z_h & \text{if } \text{out}(h) = \text{in}(h) = i, \text{ where } z_h: U \rightarrow W. \end{cases}$$

Given $(y_h, z_h)_h$ as above, the conditions for B to belong to $\mathcal{N}_{\alpha+l\alpha_i}$ are as follows :

- i) There exists a flag $F = (F_0 \subset F_1 \subset F_2 \cdots)$ such that $B_h(F_i) \subset F_i$ if $h \in \overline{\Omega}^{loop}$ and $B(F_i) \subset F_{i-1}$ otherwise,
- ii) $\mu_k(B) = 0$ for all k ,
- iii) B_h is regular semisimple for $h \in \overline{\Omega}^{loop}$.

The first condition is always satisfied: we may stack the flags F', F'' of B' and B'' together; i.e., set $F_n = F'_n$ for $n \leq \dim(V_{\alpha})$ and $F_{\dim(V_{\alpha})+m} = V_{\alpha} \oplus F''_m$ for $m \leq l$. The third condition is also always fulfilled because $(B', B'') \in \mathcal{N}_{\alpha} \times^{reg} \mathcal{N}_{l\alpha_i}$. It remains to verify the second condition, which reduces to $\mu_i(B) = 0$. At this point we distinguish two cases.

Case 1) The vertex i is real (i.e., $c_{ii} = 0$).

Since $\mu_i(B') = 0$, the moment map condition $\mu_i(B) = 0$ reads

$$(3.3) \quad 0 = \sum_{h: i \rightarrow j} \varepsilon(h)(B'_h y_h + B'_h B'_h) = \sum_{h: j \rightarrow i} \varepsilon(h) B'_h y_h.$$

This implies $\text{Im}(y: U \rightarrow \bigoplus_{h: i \rightarrow j} (V_\alpha)_j)$ lies in the kernel of the map

$$\sum_{h: i \rightarrow j} \varepsilon(h) B'_h: \bigoplus (V_\alpha)_j \rightarrow W.$$

But since $B' \in \mathcal{N}_{\alpha, i, 0}$, we have

$$\dim \text{Im} \left(\sum_{h: i \rightarrow j} \varepsilon(h) B'_h \right) = \dim(W_i) = d_i$$

and hence

$$\dim \text{Ker} \left(\sum_{h: i \rightarrow j} \varepsilon(h) B'_h \right) = \dim \left(\bigoplus_{h: i \rightarrow j} (V_\alpha)_j \right) - d_i = \sum_{j \neq i} c_{ij} d_j - d_i.$$

It follows that the fiber of p' is an affine space of dimension $l(\sum_{j \neq i} c_{ij} d_j - d_i)$ as wanted.

Case 2) The vertex i is imaginary (i.e., $c_{ii} > 0$).

Since $\mu_i(B') = \mu_i(B'') = 0$, the moment map condition $\mu_i(B) = 0$ reads

$$(3.4) \quad \begin{aligned} 0 &= \sum_{h \in \Omega_i^{\text{loop}}} ([B'_h, B'_h] + [B''_h, B''_h] + (z'_h B''_h - B'_h z'_h) + (B'_h z_h - z_h B''_h)) \\ &\quad + \sum_{\substack{h: i \rightarrow j \\ j \neq i}} \varepsilon(h)(B'_h y_h + B'_h B'_h) \\ &= \sum_{\substack{h: i \rightarrow j \\ j \neq i}} \varepsilon(h)(B'_h y_h) + \sum_{h \in \Omega_i^{\text{loop}}} ((z'_h B''_h - B'_h z'_h) + (B'_h z_h - z_h B''_h)). \end{aligned}$$

Observe that, because $B''_h \in \text{End}(U)$ and $B'_h \in \text{End}(W)$ have disjoint spectrum, the map

$$\text{Hom}(U, W) \rightarrow \text{Hom}(U, W), \quad z_h \mapsto B'_h z_h - z_h B''_h$$

is invertible for all $h \in \Omega_i^{\text{loop}}$. In particular, we may choose $(y_h)_h, (z'_h)_h$ arbitrarily as well as all z_h except for one, and uniquely solve (3.4) for that last z_h . Thus the space of solutions to (3.4) is of dimension $l(\sum_{j \neq i} c_{ij} d_j + (c_{ii} - 1)d_i)$, which completes the proof.

□

Corollary 3.2. *For any $\alpha \in Q_+$, the variety \mathcal{N}_α is Lagrangian.*

Proof. We argue by induction on α . The statement is true for $\alpha = l\alpha_i$ for some $i \in I$ and $l \in \mathbf{N}$ by Lemma 2.2. Now let $\alpha \in Q_+$ and let Λ be an irreducible component of \mathcal{N}_α . By Lemma 2.3, there exists $i \in I$ such that $\varepsilon_i(\Lambda) > 0$. Set $\varepsilon_i(\Lambda) = l$. Thus $\Lambda \cap \mathcal{N}_{\alpha, l, i}$ is open and dense in Λ . Put $\beta = \alpha - l\alpha_i$ and write $\beta = \sum_k d_k \alpha_k$.

By Proposition 3.1 (a), $q^{-1}(\Lambda \cap \mathcal{N}_{\alpha, l, i})$ is an irreducible component of $\mathcal{N}_{\beta, l, i; l\alpha_i}$ of dimension $\dim \Lambda + \dim(GL_\beta \times GL_{l\alpha_i})$. Similarly, by Proposition 3.1 (b), p is a smooth map with fibers $Z_{\beta, l\alpha_i} \times \mathbb{A}^r$ with $r = (l\alpha_i, (\text{Id} - A) \cdot \beta)$, and thus $pq^{-1}(\Lambda \cap \mathcal{N}_{\alpha, l, i})$ is an irreducible component of $\mathcal{N}_{\beta, 0, i} \times^{reg} \mathcal{N}_{l\alpha_i}$ of dimension

$$(3.5) \quad \dim \Lambda + \dim(GL_\beta \times GL_{l\alpha_i}) - \dim Z_{\beta, l\alpha_i} - r = \dim \Lambda + (l\alpha_i, (A - 2\text{Id}) \cdot \beta).$$

Recall that $\mathcal{N}_{\beta, 0, i}$ is open in \mathcal{N}_β . Hence, by the induction hypothesis, any irreducible component of $\mathcal{N}_{\beta, 0, i} \times^{reg} \mathcal{N}_{l\alpha_i}$ is of dimension

$$(3.6) \quad \frac{1}{2}(\beta, (2\text{Id} - A) \cdot \beta) + \frac{1}{2}(l\alpha_i, (2\text{Id} - A) \cdot l\alpha_i).$$

Combining (3.5) and (3.6), we get the dimension formula $\dim \Lambda = \frac{1}{2}(\alpha, (2\text{Id} - A) \cdot \alpha)$ as wanted. \square

Corollary 3.3. *For $\alpha \in Q_+$, $i \in I$ and $l \in \mathbf{N}$, there is a one-to-one correspondence between the set of irreducible components of \mathcal{N}_α satisfying $\varepsilon_i(\Lambda) = l$ and the set of irreducible components of $\mathcal{N}_{\alpha - l\alpha_i, 0, i}$.*

Proof. By Proposition 3.1, the maps p and q in (3.1), when restricted to $\mathcal{N}_{\alpha, l, i; l\alpha_i}$, are locally trivial, smooth and with connected fibers. It follows that there is a natural bijection between the sets of irreducible components of $\mathcal{N}_{\alpha, l, i}$ and $\mathcal{N}_{\alpha - l\alpha_i, 0, i} \times^{reg} \mathcal{N}_{l\alpha_i}$. By Lemma 2.2, $\mathcal{N}_{l\alpha_i}$ is irreducible, and hence we obtain a bijection between the sets of irreducible components of $\mathcal{N}_{\alpha, l, i}$ and $\mathcal{N}_{\alpha - l\alpha_i, 0, i}$. By Corollary 3.2, any irreducible component of \mathcal{N}_α is half dimensional and the same is true for all irreducible components of $\mathcal{N}_{\alpha, l, i}$. It follows that the irreducible components of $\mathcal{N}_{\alpha, l, i}$ are precisely the intersections of $\mathcal{N}_{\alpha, l, i}$ with the irreducible components of \mathcal{N}_α satisfying $\varepsilon_i(\Lambda) = l$, and we are done. \square

Following [11] and [14], we will denote this one-to-one correspondence by $\Lambda \mapsto \tilde{e}_i^l(\Lambda)$, and define the Kashiwara operators \tilde{e}_i, \tilde{f}_i on the set $\bigsqcup_{\alpha} \text{Irr } \mathcal{N}_{\alpha} \cup \{0\}$ by

$$(3.7) \quad \begin{aligned} \tilde{e}_i(\Lambda) &= \begin{cases} (\tilde{e}_i^{l-1})^{-1} \circ \tilde{e}_i^l(\Lambda) & \text{if } \varepsilon_i(\Lambda) = l > 0, \\ 0 & \text{if } \varepsilon_i(\Lambda) = 0, \end{cases} \\ \tilde{f}_i(\Lambda) &= (\tilde{e}_i^{l+1})^{-1} \circ \tilde{e}_i^l(\Lambda) \quad \text{if } \varepsilon_i(\Lambda) = l. \end{aligned}$$

Recall that we have fixed an identification $V_i \xrightarrow{\sim} V_i^*$ and defined

$$B^* = (B_h^t)_{h \in H}, \quad \varepsilon_i^*(B) = \varepsilon_i(B^*).$$

We set

$$(3.8) \quad \tilde{f}_i^* = * \circ \tilde{f}_i \circ *, \quad \tilde{e}_i^* = * \circ \tilde{e}_i \circ *.$$

The following Proposition is straightforward.

Proposition 3.4.

(a) *For any $\Lambda \in \text{Irr } \mathcal{N}_{\alpha}$, we have*

$$\tilde{e}_i \tilde{f}_i(\Lambda) = \Lambda, \quad \varepsilon_i(\tilde{f}_i(\Lambda)) = \varepsilon_i(\Lambda) + 1,$$

and if $\varepsilon_i(\Lambda) > 0$, then

$$\tilde{f}_i \tilde{e}_i(\Lambda) = \Lambda, \quad \varepsilon_i(\tilde{e}_i(\Lambda)) = \varepsilon_i(\Lambda) - 1.$$

(b) *For any $\Lambda \in \text{Irr } \mathcal{N}_{\alpha}$, we have*

$$\tilde{e}_i^* \tilde{f}_i^*(\Lambda) = \Lambda, \quad \varepsilon_i^*(\tilde{f}_i^*(\Lambda)) = \varepsilon_i^*(\Lambda) + 1,$$

and if $\varepsilon_i^(\Lambda) > 0$, then*

$$\tilde{f}_i^* \tilde{e}_i^*(\Lambda) = \Lambda, \quad \varepsilon_i^*(\tilde{e}_i^*(\Lambda)) = \varepsilon_i^*(\Lambda) - 1.$$

□

Let

$$\mathcal{B} = \coprod_{\alpha \in Q_+} \mathcal{B}_{-\alpha} = \coprod_{\alpha \in Q_+} \text{Irr } \mathcal{N}_{\alpha}.$$

For $\Lambda \in \text{Irr } \mathcal{N}_\alpha$, we define

$$\begin{aligned} \text{wt}(\Lambda) &= -\alpha, \\ \varepsilon'_i(\Lambda) &= \begin{cases} \varepsilon_i(\Lambda) & \text{if } i \in I^{re}, \\ 0 & \text{if } i \in I^{im}, \end{cases} \\ \varphi_i(\Lambda) &= \langle h_i, \text{wt}(\Lambda) \rangle + \varepsilon'_i(\Lambda). \end{aligned}$$

Then using Proposition 3.4, we obtain:

Theorem 3.5. *The maps $\text{wt}, \varepsilon'_i, \varphi_i, \tilde{f}_i, \tilde{e}_i$ ($i \in I$) define a $U_q(\mathfrak{g})$ -crystal structure on \mathcal{B} . \square*

To finish this section, we introduce the following useful notation. Let $\Lambda \in \text{Irr } \mathcal{N}_\alpha$ and put $l = \varepsilon_i(\Lambda)$. Let B be a generic element of Λ so that $\varepsilon_i(B) = l$. We define the element $B' = \tilde{e}_i^l(B)$ as follows. Let $W \subset V_i$ be the characteristic subspace

$$W = \mathbf{C}\langle B \rangle_i \cdot \sum_{\substack{h: j \rightarrow i \\ j \neq i}} \text{Im } B_h.$$

It is of dimension $d_i - l$, where $\alpha = \sum_k d_k \alpha_k$. Then $B' \in \mathcal{N}_{\alpha - l\alpha_i}$ is the restriction of B to the subspace $V' \subset V$, where $V'_j = V_j$ for $j \neq i$ and $V'_i = W$. Moreover, it is a generic element of $\tilde{e}_i^l(\Lambda)$. Of course, a similar definition can be given for \tilde{e}_i^* as well.

4. GEOMETRIC CONSTRUCTION OF $B(\infty)$

Fix $i \in I$ and let $B_i = \{b_i(-n); n \geq 0\}$ be the elementary crystal. We define a map $\Psi_i: \mathcal{B} \longrightarrow \mathcal{B} \otimes B_i$ by

$$(4.1) \quad \Psi_i(\Lambda) = \tilde{e}_i^{*c} \Lambda \otimes b_i(-c),$$

where $c = \varepsilon_i^*(\Lambda)$.

Theorem 4.1. *The map $\Psi_i: \mathcal{B} \longrightarrow \mathcal{B} \otimes B_i$ is a strict crystal embedding.*

Proof. It is clear that the underlying map is injective. We will prove

$$(4.2) \quad \Psi_i(\tilde{e}_j \Lambda) = \tilde{e}_j \Psi_i(\Lambda) \quad \text{for all } j \in I.$$

We distinguish several cases. If $i \in I^{re}$, then the proof of [11] goes through with no modification. So we assume that $i \in I^{im}$.

Case 1) $i \neq j$.

Since $\varepsilon'_j(b_i(-c)) = -\infty$, the tensor product rule yields

$$\tilde{e}_j \Psi_i(\Lambda) = \tilde{e}_j(\tilde{e}_i^{*c} \Lambda \otimes b_i(-c)) = \tilde{e}_j \tilde{e}_i^{*c} \Lambda \otimes b_i(-c).$$

On the other hand, we have

$$\Psi_i(\tilde{e}_j \Lambda) = \tilde{e}_i^{*d}(\tilde{e}_j \Lambda) \otimes b_i(-d),$$

where $d = \varepsilon_i^*(\tilde{e}_j \Lambda)$. Assume that $\tilde{e}_j(\Lambda) = 0$; i.e., $\varepsilon_j(\Lambda) = 0$. We claim $\varepsilon_j(\tilde{e}_i^{*c}(\Lambda)) = 0$ as well so that $\tilde{e}_j(\tilde{e}_i^{*c}(\Lambda)) = 0$. Indeed, we have

Lemma 4.2. *If $i \neq j$, then $\varepsilon_j(\tilde{e}_i^*(\Lambda)) = \varepsilon_j(\Lambda)$ for every Λ .*

Proof. It is enough to see that $\varepsilon_j(\tilde{e}_i^{*l} \Lambda) = \varepsilon_j(\Lambda)$ if $\varepsilon_i^*(\Lambda) = l$. Let B be a generic point of Λ and let $B' = p_1 q^{-1}(B)$ be the corresponding generic point of $\tilde{e}_i^{*l}(\Lambda)$ (see (3.2)). Then for any edge $h \in H$ with $\text{in}(h) \neq i$, we have $\text{Im}(B_h) = \text{Im}(B'_h)$. In particular,

$$\mathbf{C}\langle B \rangle_i \cdot \sum_{\substack{h: k \rightarrow j \\ k \neq j}} \text{Im } B_h = \mathbf{C}\langle B' \rangle_i \cdot \sum_{\substack{h: k \rightarrow j \\ k \neq j}} \text{Im } B'_h,$$

which gives the desired equality. \square

Now assume that $\varepsilon_j(\Lambda) > 0$. To prove (4.2), thanks to the above lemma, we have only to show that $\tilde{e}_j \tilde{e}_i^{*c}(\Lambda) = \tilde{e}_i^{*c} \tilde{e}_j(\Lambda)$, which is a direct consequence of the following Lemma.

Lemma 4.3. *If $i \neq j$, then $\tilde{e}_j \tilde{e}_i^*(\Lambda) = \tilde{e}_i^* \tilde{e}_j(\Lambda)$ for every Λ .*

Proof. Put $a = \varepsilon_j(\Lambda) = \varepsilon_j(\tilde{e}_i^*(\Lambda))$. It is enough to show that $\tilde{e}_j^a \tilde{e}_i^*(\Lambda) = \tilde{e}_i^* \tilde{e}_j^a(\Lambda)$ for all Λ . Indeed, we then have $\tilde{e}_j^{a-1} \tilde{e}_i^* \tilde{e}_j(\Lambda) = \tilde{e}_i^* \tilde{e}_j^{a-1} \tilde{e}_j(\Lambda) = \tilde{e}_i^* \tilde{e}_j^a(\Lambda) = \tilde{e}_j^a \tilde{e}_i^*(\Lambda) = \tilde{e}_j^{a-1} \tilde{e}_j \tilde{e}_i^*(\Lambda)$ from which we deduce $\tilde{e}_i^* \tilde{e}_j(\Lambda) = \tilde{e}_j \tilde{e}_i^*(\Lambda)$. Similarly, it is enough to prove that $\tilde{e}_i^{*b} \tilde{e}_j^a(\Lambda) = \tilde{e}_j^a \tilde{e}_i^{*b}(\Lambda)$, where $b = \varepsilon_i^*(\Lambda) = \varepsilon_i^*(\tilde{e}_j^a(\Lambda))$. This can be done by chasing the diagram given in the following, where $B \in \Lambda$ is a generic point, $\sigma = \alpha - a\alpha_j$, $\beta = \alpha - b\alpha_i$, and the middle horizontal line (resp. the middle vertical line) represents the short exact sequence defining $\tilde{e}_i^{*b}(\Lambda)$ (resp. $\tilde{e}_j^a(\Lambda)$).

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & V_{b\alpha_i} & \longrightarrow & V_\sigma & \longrightarrow & \text{Im}(V_\sigma \rightarrow V_\beta) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & V_{b\alpha_i} & \longrightarrow & V_\alpha & \longrightarrow & V_\beta \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & V_{a\alpha_j} & \xlongequal{\quad\quad\quad} & V_{a\alpha_j} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Note that V_β and V_σ are uniquely determined by B , and that the right column and the top row represent $\tilde{e}_j^a \tilde{e}_i^{*b}(\Lambda)$ and $\tilde{e}_i^{*b} \tilde{e}_j^a(\Lambda)$, respectively. \square

Case 2) $i = j$.

Write

$$\begin{aligned}
 \text{wt}(\Lambda) &= -\alpha = -m\alpha_i - \sum_{k \neq i} m_k \alpha_k, \\
 \text{wt}(\tilde{e}_i^{*c}(\Lambda)) &= -\alpha + c\alpha_i = -(m-c)\alpha_i - \sum_{k \neq i} m_k \alpha_k.
 \end{aligned}$$

Thus

$$\begin{aligned}
 \varphi_i(\tilde{e}_i^{*c}(\Lambda)) &= (m-c)(-a_{ii}) + \sum_{k \neq i} m_k(-a_{ik}), \\
 \varepsilon'_i(b_i(-c)) &= 0.
 \end{aligned}$$

To prove our claim, we consider the following three cases:

- (a) $\varphi_i(\tilde{e}_i^{*c}(\Lambda)) \leq 0$,
- (b) $0 < \varphi_i(\tilde{e}_i^{*c}(\Lambda)) \leq -a_{ii}$,
- (c) $-a_{ii} < \varphi_i(\tilde{e}_i^{*c}(\Lambda))$.

(a) The condition (a) implies that $(m-c)(-a_{ii}) = 0$ and $m_k(-a_{ik}) = 0$ for all $k \neq i$. Hence $\dim V_k = 0$ whenever there is an arrow $h: i \rightarrow k$ ($k \neq i$), which implies $B_h = B_h^t = 0$ for all such h . Hence

$$c = \varepsilon_i^*(\Lambda) = \text{codim}_{V_i} 0 = \dim V_i = m.$$

For similar dimension reasons, we have $\varepsilon_i^*(\tilde{e}_i(\Lambda)) = c - 1$. Moreover, for any $l \geq 1$, our Lagrangian varieties decompose as

$$\mathcal{N}_{\alpha-l\alpha_i} \simeq \mathcal{N}_{(m-l)\alpha_i} \times \mathcal{N}_{\alpha'},$$

where $\alpha' = \sum_{k \neq i} m_k \alpha_k$. The Kashiwara operators $\tilde{e}_i, \tilde{e}_i^*$ act on the first component of this decomposition. By Lemma 2.2, we have $\text{Irr } \mathcal{N}_{l\alpha_i} = \{pt\}$ for all l , and it is clear that $\tilde{e}_i^* = \tilde{e}_i: \text{Irr } \mathcal{N}_{l\alpha_i} \xrightarrow{\sim} \text{Irr } \mathcal{N}_{(l-1)\alpha_i}$ for all l . In particular, $\tilde{e}_i^{*(c-1)}\tilde{e}_i(\Lambda) = \tilde{e}_i^{*c}(\Lambda)$ and therefore

$$\begin{aligned} \tilde{e}_i \Psi_i(\Lambda) &= \tilde{e}_i(\tilde{e}_i^{*c} \Lambda \otimes b_i(-c)) = \tilde{e}_i^{*c} \Lambda \otimes b_i(-c+1) \\ &= \tilde{e}_i^{*(c-1)}(\tilde{e}_i \Lambda) \otimes b_i(-c+1) = \Psi_i(\tilde{e}_i \Lambda), \end{aligned}$$

which proves our claim in the case (a).

(b) In this case, we have $a_{ii} < 0$. We already know $m \geq c$. The condition (b) implies

$$(m - c - 1)a_{ii} \geq \sum_{k \neq i} m_k(-a_{ik}) \geq 0.$$

Since $a_{ii} < 0$, we must have $m \leq c + 1$; i.e., $m = c$ or $c + 1$.

If $m = c + 1$, we have $m_k = \dim V_k = 0$ whenever $a_{ik} \neq 0$ ($k \neq i$), and hence

$$c = \varepsilon_i^*(B) = \text{codim}_{V_i} \mathbf{C}\langle B^t \rangle_i = \sum_{h: i \rightarrow k, k \neq i} \text{Im } B_h^t = \text{codim}_{V_i} 0 = m = c + 1,$$

which is a contradiction.

Hence $m = c$ and $\sum_{k \neq i} m_k(-a_{ik}) > 0$. By the definition, $m = c$ means that $B_h = 0$ for all $h: i \rightarrow k, k \neq i$. Moreover, there exists k such that $m_k > 0$ and $a_{ik} \neq 0$. We claim that under these conditions $\varepsilon_i(\Lambda) = 0$. Indeed, since $B_h = 0$ for all $h: i \rightarrow k$ on Λ , which is coisotropic, all $B_{\bar{h}}$ for $\bar{h}: k \rightarrow i$ may be chosen arbitrarily. But because B_σ is regular semisimple for any $\sigma \in \overline{\Omega}_i^{\text{loop}}$, for a generic $B \in \Lambda$, we can choose $B_{\bar{h}}$ such that $\mathbf{C}\langle B \rangle_i \text{Im } B_{\bar{h}} = V_i$. It follows that

$$\mathbf{C}\langle B \rangle_i \cdot \sum_{\substack{h: k \rightarrow i \\ k \neq i}} \text{Im } B_h = V_i$$

as wanted. Hence $\tilde{e}_i(\Lambda) = 0$ and by the tensor product rule, we obtain

$$\tilde{e}_i \Psi_i(\Lambda) = \tilde{e}_i(\tilde{e}_i^{*c} \Lambda \otimes b_i(-c)) = 0 = \Psi_i(\tilde{e}_i \Lambda).$$

(c) If $m = c$, the condition (c) implies there exists $k \neq i$ such that $m_k > 0$, $a_{ik} \neq 0$. By the same argument in (b), one can deduce $\tilde{e}_i \Lambda = 0$. On the other hand, since $m = c$, we have $\dim_i(\tilde{e}_i^{*c}(\Lambda)) = 0$ and therefore $\varepsilon_i(\tilde{e}_i^{*c} \Lambda) = 0$, which implies $\tilde{e}_i(\tilde{e}_i^{*c} \Lambda) = 0$. Hence by the tensor product rule, we have

$$\begin{aligned} \tilde{e}_i \Psi_i(\Lambda) &= \tilde{e}_i(\tilde{e}_i^{*c} \Lambda \otimes b_i(-c)) = \tilde{e}_i(\tilde{e}_i^{*c} \Lambda) \otimes b_i(-c) \\ &= 0 \otimes b_i(-c) = 0 = \Psi_i(\tilde{e}_i \Lambda). \end{aligned}$$

Let us now assume that $m > c$. By the tensor product rule again, we have to prove that $\tilde{e}_i^{*c} \tilde{e}_i(\Lambda) = \tilde{e}_i \tilde{e}_i^{*c}(\Lambda)$ and $\varepsilon_i^*(\tilde{e}_i(\Lambda)) = c$. Let B be a generic element of Λ and let us consider the characteristic spaces

$$W = \mathbf{C}\langle B \rangle_i \cdot \sum_{\substack{h: j \rightarrow i \\ j \neq i}} \text{Im } B_h,$$

$$U = \bigcap_{\substack{h: i \rightarrow j \\ j \neq i}} \text{Ker}(B_h \cdot \mathbf{C}\langle B \rangle_i).$$

Then we have $\dim U = \varepsilon_i^*(B) = c$ and $\text{codim}_V W = \varepsilon_i(B)$.

We first claim $U \subset W$. Let $d = \varepsilon_i(\Lambda)$ and put $B' = \tilde{e}_i^d(B)$. The operator B' acts on the subspace $V' \subset V$ with $V'_k = V_k$ for $k \neq i$ and $V'_i = W$. Moreover $B|_{V'} = B'$ and B can be viewed as a (generic) element in the fiber of $\{B'\} \times \mathcal{N}_{d\alpha_i}$ under the map $B \mapsto (B|_{V'}, B|_{V/V'})$. Take $\sigma \in \overline{\Omega}_i^{\text{loop}}$ so that B_σ is regular semisimple. Since B_σ preserves W , we may choose a splitting $V_i = W \oplus T$ invariant under B_σ . Let $\{v_1, \dots, v_t\}$ be the basis of T consisting of B_σ -eigenvectors. Since we have assumed $m > c$, $U \neq V_i$, and there exists $k \neq i$ such that $a_{ik} \neq 0$ and $V_k \neq \{0\}$. Let $h: i \rightarrow k$ be an edge in H . Since $B_h|_T: T \rightarrow V_k$ may be chosen arbitrarily (see the proof of Proposition 3.1), $B_h(v_l) \neq 0$ ($l = 1, \dots, t$) for generic B . Hence U does not contain v_1, \dots, v_t . Since U is invariant under B_σ , $U \subset W$.

By the above claim, we have $d := \varepsilon_i(\Lambda) = \varepsilon_i(\tilde{e}_i^{*c}(\Lambda))$ and for a generic $B \in \Lambda$, $B' := \tilde{e}_i^d \tilde{e}_i^{*c}(B)$ is the operator induced by B on the space $\bigoplus_{j \neq i} V_j \oplus W/U$. On the the hand, we have $\varepsilon_i^*(\tilde{e}_i(\Lambda)) = \dim U = c$ and $\varepsilon_i(\tilde{e}_i^{*c} \tilde{e}_i(\Lambda)) = \varepsilon_i(\Lambda) - 1$. It is easy to see that $B'' := \tilde{e}_i^{d-1} \tilde{e}_i^{*c} \tilde{e}_i(B)$ coincides with B' and hence that $\tilde{e}_i^{*c} \tilde{e}_i(\Lambda) = \tilde{e}_i \tilde{e}_i^{*c}(\Lambda)$. We are done with case (c), and hence with the proof of Theorem 4.1 \square

Now we obtain the main result of this paper.

Theorem 4.4. *There exists a crystal isomorphism*

$$\mathcal{B} = \coprod_{\alpha \in Q_+} \text{Irr } \mathcal{N}_\alpha \xrightarrow{\sim} B(\infty).$$

Proof. Our assertion follows from Theorem 1.7, Lemma 2.3, Theorem 3.5, and Theorem 4.1. \square

Example 4.5.

(a) Let (I, H) be the quiver with one vertex $I = \{i\}$ and $2t$ edge loops in H . Thus there are t edge loops in Ω and $\overline{\Omega}$, respectively, and the corresponding Borcherds-Cartan matrix is $A = (2 - 2t)$.

For $l \geq 1$, let $V = \mathbf{C}^l$ be the I -graded vector space with $\underline{\dim} V = l\alpha_i$. By Lemma 2.2, the variety $\mathcal{N}_{l\alpha_i}$ is irreducible and Lagrangian. Moreover, if $B = (B_i, \overline{B}_i)_{1 \leq i \leq t}$ is a generic element of $\mathcal{N}_{l\alpha_i}$, then we have

$$\begin{aligned} \varepsilon_i(B) &= \text{codim}_{V_i} 0 = l, \\ \varepsilon_i^*(B) &= \varepsilon_i(B^t) = \text{codim}_{V_i} 0 = l. \end{aligned}$$

Hence we have

$$\Psi_i(\tilde{f}_i^l \mathbf{1}) = \mathbf{1} \otimes b_i(-l),$$

and the crystal structure on $B(\infty)$ is given as follows.

$$\mathbf{1} \xrightarrow{i} \tilde{f}_i \mathbf{1} = \mathcal{N}_{\alpha_i} \xrightarrow{i} \cdots \xrightarrow{i} \tilde{f}_i^l \mathbf{1} = \mathcal{N}_{l\alpha_i} \xrightarrow{i} \cdots$$

(b) Let (I, H) be a quiver, where $I = \{i, j\}$ and H consists of two edge loops at i , one arrow from i to j , and one arrow from j to i . Thus the corresponding Borcherds-Cartan matrix is $A = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$. We choose an orientation Ω consisting of an edge loop at i and an arrow from i to j .

We will compute $\tilde{f}_j \tilde{f}_i^2 \mathbf{1}$. Let $V = V_i \oplus V_j = \mathbf{C}^2 \oplus \mathbf{C}$ be the I -graded vector space with $\underline{\dim} V = 2\alpha_i + \alpha_j$, and let $B = (B_1, \overline{B}_1, B_2, \overline{B}_2)$ be a generic element in $\mathcal{N}_{2\alpha_i + \alpha_j}$, where $B_1, \overline{B}_1: \mathbf{C}^2 \rightarrow \mathbf{C}^2$, $B_2: \mathbf{C}^2 \rightarrow \mathbf{C}$, $\overline{B}_2: \mathbf{C} \rightarrow \mathbf{C}^2$ are the linear maps satisfying the conditions for $\mathcal{N}_{2\alpha_i + \alpha_j}$. By applying the Kashiwara operators successively to $\mathbf{1}$, one can deduce that, for an appropriate basis of V , B has the form

$$B = (B_1, \overline{B}_1, B_2, \overline{B}_2) = \left(0, \begin{pmatrix} \lambda & a \\ 0 & \mu \end{pmatrix}, 0, \begin{pmatrix} x \\ y \end{pmatrix} \right),$$

where $a, \lambda, \mu, x, y \in \mathbf{C}$, $\lambda \neq \mu$. Note that

$$\begin{aligned}\varepsilon_i(B) &= \text{codim}_{V_i} \mathbf{C}\langle B_1, \overline{B_1} \rangle_i \text{Im} \overline{B_2} = 0, \\ \varepsilon_i^*(B) &= \varepsilon_i(B^t) = \text{codim}_{V_i} \mathbf{C}\langle B_1^t, \overline{B_1^t} \rangle_i 0 = 2, \\ \varepsilon_j(B) &= \text{codim}_{V_j} \text{Im} B_2 = \text{codim}_{V_j} 0 = 1, \\ \varepsilon_j^*(B) &= \text{codim}_{V_j} \text{Im} \overline{B_2^t} = 0.\end{aligned}$$

Hence we have

$$\begin{aligned}\Psi_i(\tilde{f}_j \tilde{f}_i^2 \mathbf{1}) &= \tilde{e}_i^{*2}(\tilde{f}_j \tilde{f}_i^2 \mathbf{1}) \otimes b_i(-2) = \tilde{f}_j \mathbf{1} \otimes b_i(-2), \\ \Psi_j(\tilde{f}_j \tilde{f}_i^2 \mathbf{1}) &= \tilde{f}_j \tilde{f}_i^2 \mathbf{1} \otimes b_j(0).\end{aligned}$$

□

REFERENCES

- [1] R. E. Borcherds, *Generalized Kac-Moody algebras*, J. Algebra **115** (1988), 501–512.
- [2] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405–444.
- [3] J. Fuchs, U. Ray, C. Schweigert, *Some automorphisms of generalized Kac-Moody algebras*, J. Algebra **191** (1997), 518–540.
- [4] K. Jeong, S.-J. Kang, M. Kashiwara, *Crystal bases for quantum generalized Kac-Moody algebras*, Proc. Lond. Math. Soc. (3) **90** (2005), 395–438.
- [5] K. Jeong, S.-J. Kang, M. Kashiwara, D.-U. Shin, *Abstract crystals for quantum generalized Kac-Moody algebras*, Intern. Math. Res. Notices **2007-001** (2007), 1–18.
- [6] S.-J. Kang, *Quantum deformations of generalized Kac-Moody algebras and their modules*, J. Algebra **175** (1995), 1041–1066.
- [7] S.-J. Kang, J.-H. Kwon, *Graded Lie superalgebras, supertrace formula, and orbit Lie superalgebras*, Proc. London Math. Soc. **81** (2000), 675–724.
- [8] S.-J. Kang, O. Schiffmann, *Canonical bases for generalized Kac-Moody algebras*, Adv. in Math. **200**, (2006), 455–478.
- [9] M. Kashiwara, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
- [10] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften **292**, Springer-Verlag Berlin Heidelberg, 1990.
- [11] M. Kashiwara and Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), 9–36.
- [12] Y. Li, Z. Lin, *Canonical bases of Cartan-Borcherds type*, preprint (2007).

- [13] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), no. 2, 365–421.
- [14] G. Lusztig, *Affine quivers and canonical bases*, Inst. Hautes Études Sci. Publ. Math. No. **76** (1992), 111–163.
- [15] G. Lusztig, *Tight monomials in quantized enveloping algebras*, in *Quantum deformations of algebras and their representations*, Isr. Math. Conf. Proc. **7** (1993), Amer. Math. Soc. 117–132.
- [16] S. Naito, *Kazhdan-Lusztig conjecture for generalized Kac-Moody algebras II: Proof of the conjecture*, Trans. Amer. Math. Soc. **347** (1995), 3891–3920.
- [17] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke math. J. **76** (1994), no. 2, 365–416.
- [18] C. M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), no. 3, 583–591.
- [19] Y. Saito, *Crystal bases and quiver varieties*, Math. Ann. **324** (2002), no. 4, 675–688.

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